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Higher Algebraic K-theory for Twisted Laurent Series Rings Over Orders and Semisimple Algebras

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Abstract Let R be the ring of integers in a number field F, Λ any R-order in a semisimple F-algebra Σ, α an R-automorphism of Λ. Denote the extension of α to Σ also by α. Let $\Lambda_\alpha[T]$ (resp. $\Sigma_\alpha[T]$ be the α-twisted Laurent series ring over Λ (resp. Σ). In this paper we prove that (i) There exist isomorphisms $\mathbb{Q} \otimes K_n(\Lambda_\alpha[T]) \simeq \mathbb{Q} \otimes G_n(\Lambda_\alpha[T]) \simeq \mathbb{Q} \otimes K_n(\Sigma_\alpha[T])$ for all $n \geq 1$. (ii) $G_n^{\mathrm{pr}}(\Lambda_\alpha[T], \hat{Z}_l) \simeq G_n(\Lambda_\alpha[T], \hat{Z}_l)$ is an l-complete profinite Abelian group for all $n \geq 2$. (iii) $\operatorname{div} G_n^{\mathrm{pr}}(\Lambda_\alpha[T], \hat{Z}_l) = 0$ for all $n \geq 2$. (iv) $G_n(\Lambda_\alpha[T]) \longrightarrow G_n^{\mathrm{pr}}(\Lambda_\alpha[T], \hat{Z}_l)$ is injective with uniquely l-divisible cokernel (for all $n \geq 2$). (v) $K_{-1}(\Lambda)$, $K_{-1}(\Lambda_\alpha[T])$ are finitely generated Abelian groups.

Keywords K-theory · Twisted Laurent series rings · Semisimple algebras · Orders · Virtually infinite cyclic group

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1. Introduction

Let R be the ring of integers in a number field F. The initial motivation for this work was a desire to obtain results on higher K-theory of the groupring RV of a virtually infinite cyclic group of the form $V = G \rtimes_\alpha T$, where G is a finite group, α

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an automorphism of G and the action of the infinite cyclic group $T = \langle t \rangle$ on G is given by $\alpha(g) = tgt^{-1}$ for all $g \in G$.

Note that understanding the K-theory of RV is fundamental to the Farrell-Jones conjecture which asserts that K-theory of an arbitrary discrete group H should have as "building blocks" the K-theory of virtually cyclic subgroups of H (see [8]). A group V is virtually cyclic if it is either finite or virtually infinite cyclic (i.e., contains a finite index subgroup that is infinite cyclic). For results on higher K-theory of grouprings of finite groups see [15, chapter 7] and associated references. There are two types of virtually infinite cyclic groups — one type of the form $V = G \rtimes_\alpha T$ as described above and the other of the form $V = G_0 *_H G_1$, where the groups G_0, G_1, H are finite and $[G_0 : H] = [G_1 : H] = 2$. For some results on higher K-theory of both types of groups see [15, 7.5] or [16]. In this paper, we obtain results on higher K-theory of twisted Laurent series ring that translate into results on grouprings RV, $V = G \rtimes_\alpha T$, as we now explain.

If α is an automorphism of a finite group G, we also denote by α the automorphism induced on RG by α and observe that for $V = G_\alpha \rtimes T$, $RV = (RG)_\alpha[T] = (RG)_\alpha[t, t^{-1}]$ is the α-twisted Laurent series ring over the groupring RG. Now, RG is an R-order in the semi-simple F-algebra FG and so, we endeavour in this paper to obtain general results on higher K-theory of $\Lambda_\alpha(T)$ where Λ is an arbitrary R-order in a semi-simple F-algebra Σ so that results on $(RG)_\alpha[T]$ become examples and applications of our results.

Note also that an R-automorphism of Λ extends to an F-automorphism of Σ which we also denote by α. We also study higher K-theory of $\Sigma_\alpha[T]$ and prove in Theorem 1(b) that there exist isomorphisms

$$\mathbb{Q} \otimes K_n(\Lambda_\alpha[T]) \simeq \mathbb{Q} \otimes G_n(\Lambda_\alpha[T]) \simeq \mathbb{Q} \otimes K_n(\Sigma_\alpha[T])$$

for all $n \geq 2$. Hence $\mathbb{Q} \otimes K_n(RV) \simeq \mathbb{Q} \otimes G_n(RV) \simeq Q \otimes K_n(FV)$ for all $n \geq 2$. Since we have shown in Theorem 1(a) that $G_n(\Lambda_\alpha[T])$ is a finitely generated Abelian group for all $n \geq 1$, it follows that $K_n(\Lambda_\alpha[T])$, $K_n(\Sigma_\alpha[T])$ and hence $K_n(RV)$, $K_n(FV)$ have finite torsion-free ranks for all $n \geq 2$.

We next investigate under what conditions $G_n(\Lambda_\alpha[T])$ could actually be a finite group and show in Theorem 6 that when F is a totally real number field with ring of integers R and Λ any R-order in a semi-simple F-algebra, then $G_{2(m+1)}(\Lambda_\alpha[T])$ is finite for all odd $m \geq 1$. Hence $G_{2(m+1)}(RV)$ is finite.

In Section 3, we study profinite higher K-theory of $\Lambda_\alpha[T]$ and prove that $G_n^{\mathrm{pr}}(\Lambda_\alpha[T], \hat{\mathbb{Z}}_l) = G_n(\Lambda_\alpha[T], \hat{\mathbb{Z}}_l)$ are l-complete profinite Abelian groups; $\operatorname{div} G_n^{\mathrm{pr}}(\Lambda_\alpha[T], \hat{\mathbb{Z}}_l) = 0$; and that the map $G_n(\Lambda_\alpha[T]) \longrightarrow G_n^{\mathrm{pr}}(\Lambda_\alpha[T], \hat{\mathbb{Z}}_l)$ is injective with uniquely l-divisible cokernel. Corresponding results follow when we replace $\Lambda_\alpha[T]$ by RV.

In a final section, we prove that if F is an algebraic number field with ring of integers R and Λ any R-order in a semi-simple F-algebra Σ, then $K_{-1}(\Lambda)$ and $K_{-1}(\Lambda_\alpha[T])$ are finitely generated Abelian groups; $NK_{-1}(\Lambda, \alpha) = 0$ and $K_{-1}(\Lambda[t]) \simeq K_{-1}(\Lambda)$. That $K_{-1}(\Lambda)$ and $K_{-1}(\Lambda_\alpha[T])$ are finitely generated for arbitrary R-orders Λ generalizes similar results by D. Carter for $K_{-1}(RG)$ (G a finite group, see [4]) resp. by Farrell/Jones for $K_{-1}(\mathbb{Z}V)$ (see [9]).

Notes on notation If α is an automorphism of a ring A, we shall write $A_\alpha[T] = A_\alpha[t, t^{-1}]$ for the α-twisted Laurent series ring over A. Note that additively $A_\alpha[T] =$

$A_\alpha[t, t^{-1}]$ with multiplication given by $(at^i)\cdot(bt^j) = a\alpha^{-1}(b)t^{i+j}$ for $a, b \in A$. $A_\alpha[t]$ (resp. $A_\alpha[t^{-1}]$) is the subring of $A_\alpha[T]$ generated by A and t (resp. A and t^{-1}). Call $A_\alpha[t]$ the α-twisted polynomial ring over A. We also have inclusion maps $i : A \to A_\alpha[T]$, $i^+ : A \to A_\alpha[t]$ and $i^- : A \to A_\alpha[t^{-1}]$.

The augmentation map $\varepsilon : A_\alpha[t] \to A$ induces a group homomorphism $\varepsilon_* : K_n(A_\alpha[t]) \to K_n(A)$ and we put $NK_n(A, \alpha) := \ker \varepsilon_*$. Since ε is split by i^+, we have $K_n(A_\alpha[t]) \simeq K_n(A) \oplus NK_n(A, \alpha)$.

If B is an additive Abelian group and m is a positive integer, we shall write B/m for B/mB and $B[m]$ for the set of elements x of B such that $mx = 0$. We write div B for the subgroup of divisible elements of B. If l is a rational prime, we write B_l for the l-primary subgroup of B. Note that $B_l = \bigcup B[l^s] = \varinjlim B[l^s]$.

2. Higher K-theory of $\Lambda_\alpha[T]$, $\Sigma_\alpha[T]$ (Λ Arbitrary Orders)

2.1. $K_n(\Lambda_\alpha[T])$, $G_n(\Lambda_\alpha[T])$, $K_n(\Sigma_\alpha[T])$

2.1.1 Let R be the ring of integers in a number field F, Λ any R-order in a semi-simple F-algebra Σ, α an R-automorphism of Λ. Then α can be extended to an F-automorphism of Σ (since $\Sigma = \Lambda \otimes_R F$). The aim of this section is to prove the following theorem.

Theorem 1 *Let F be an algebraic number field with ring of integers R, Λ any R-order in a semi-aimple F-algebra Σ, α an R-automorphism of Λ. Denote the extension of α to Σ also by α. Let $\Lambda_\alpha[T]$ (resp. $\Sigma_\alpha[T]$) be the α-twisted Laurent series ring over Λ (resp. Σ). Then we have*

(a) *$G_n(\Lambda_\alpha[T])$ is a finitely generated Abelian group for all $n \geq 1$.*
(b) *There exist isomorphisms:*

$$\mathbb{Q} \otimes K_n(\Lambda_\alpha[T]) \simeq \mathbb{Q} \otimes G_n(\Lambda_\alpha[T]) \simeq \mathbb{Q} \otimes K_n(\Sigma_\alpha[T])$$

for $n \geq 2$.

Before proving Theorem 1 we state the following consequence of the result.

Corollary 1 *Let $V = G \rtimes_\alpha T$ be the virtually infinite cyclic subgroup where G is a finite group, $\alpha \in \mathrm{Aut}(G)$ and the action of T on G is given by $\alpha(g) = tgt^{-1}$, for all $g \in G$. Then,*

(a) *$G_n(RV)$ is a finitely generated Abelian group for al $n \geq 1$.*
(b) *$\mathbb{Q} \otimes K_n(RV) \simeq \mathbb{Q} \otimes G_n(RV) \simeq \mathbb{Q} \otimes K_n(FV)$ for all $n \geq 2$.*

The proof of Theorem 1(b) will proceed in several steps (see Theorems 3, 4, 5 below). However, we first recall the following result: Theorem 2.

Theorem 2 ([15, Theorem 7.3.2] or [16]) *Let R be the ring of integers in a number field F, Λ any R-order in a semi-simple F-algebra Σ. If $\alpha : \Lambda \to \Lambda$ is an R-automorphism, then there exists an R-order $\Gamma \subset \Sigma$, such that*

(1) *$\Lambda \subset \Gamma$;*
(2) *Γ is α-invariant;*
(3) *Γ is (right) regular ring. In fact Γ is (right) hereditary.*

Theorem 3 *Let R be the ring of integers in a number field F, Λ any R-order in a semi-simple F-algebra, $\alpha : \Lambda \to \Lambda$ and R-automorphism of Λ, Γ an α-invariant order containing Λ as in Theorem 2, $\Lambda_\alpha[T]$ (resp. $\Gamma_\alpha[T]$) the α-twisted Laurent series ring over Λ (resp. Γ). $\varphi : \Lambda_\alpha[T] \to \Gamma_\alpha[T]$ the map induced by the inclusion $\Lambda \to \Gamma$. Then the induced homomorphisms $\varphi_n : K_n(\Lambda_\alpha[T]) \to K_n(\Gamma_\alpha[T])$ has torsion kernel and cokernel. Hence for all $n \geq 2$ we have $\mathbb{Q} \otimes K_n(\Lambda_\alpha[T]) \simeq \mathbb{Q} \otimes K_n(\Gamma_\alpha[T])$.*

Proof There exists a positive integer s such that $s\Gamma \subset \Lambda$ (see [18] or [15]). Put $q = s\Gamma$. Then $\underline{q}$ is an ideal of Γ and Λ. Put $B = \Lambda/\underline{q}$, $B' = \Gamma/\underline{q}$. Then we have cartesian squares

$$\begin{array}{ccc} \Lambda & \longrightarrow & \Gamma \\ \downarrow & & \downarrow \\ B & \longrightarrow & B' \end{array} \tag{1}$$

and

$$\begin{array}{ccc} \Lambda_\alpha[T] & \longrightarrow & \Gamma_\alpha[T] \\ \downarrow & & \downarrow \\ B_\alpha[T] & \longrightarrow & B'_\alpha[T]. \end{array} \tag{2}$$

So, by [5] and [19], we have a long exact sequence

$$\begin{aligned} \cdots \to K_{n+1}\big(B'_\alpha[T]\big)\left(\frac{1}{s}\right) &\longrightarrow K_n(\Lambda_\alpha[T])\left(\frac{1}{s}\right) \\ &\longrightarrow K_n\big(\Gamma_\alpha[T]\big)\left(\frac{1}{s}\right) \oplus K_n\big(B_\alpha[T]\big)\left(\frac{1}{s}\right) \\ &\longrightarrow K_n\big(B'_\alpha[T]\big)\left(\frac{1}{s}\right) \to \cdots \end{aligned} \tag{3}$$

Now, Γ, B, B' are quasi-regular rings, so are $\Gamma_\alpha[T]$, $B_\alpha[T]$ and $B'_\alpha[T]$ (see [9]). If we write A for $B_\alpha[T]$ or $B'_\alpha[T]$, JA for the Jacobson's radical of A, then by [19] $K_n(A, JA)$ is s-torsion since s annihilates A and so from the relative sequence

$$\cdots \longrightarrow K_n(A, JA) \longrightarrow K_n(A) \longrightarrow K_n(A/J) \longrightarrow \cdots$$

we have $K_n(A)\left(\frac{1}{s}\right) \simeq K_n(A/JA)\left(\frac{1}{s}\right)$. We now claim that $K_n(A)\left(\frac{1}{s}\right) \simeq K_n(A/JA)\left(\frac{1}{s}\right)$ is torsion.

Proof of the claim Note that $A/JA \simeq (A'/JA')_\alpha[T]$ is a regular ring (see [9]) where A'/JA' is a finite semi-simple ring which is a finite direct product of matrix algebras over finite fields. Hence $K_n((A'/JA')_\alpha[T])$ is a finite direct sum of K-groups of the form $K_n((F_i)_\alpha[T])$ where F_i is a finite field. Also, $(F_i)_\alpha[T]$ is a regular ring and so $K_n((F_i)_\alpha[T]) \simeq G_n((F_i)_\alpha[T])$. □

Now, for each F_i, we have by [15, Theorem 7.5.3(iii)] or [16], that there exists a long exact sequence

$$\cdots \to G_n(F_i) \longrightarrow G_n(F_i) \longrightarrow G_n((F_i)_\alpha[T]) \\ \longrightarrow G_{n-1}(F_i) \longrightarrow G_{n-1}(F_i) \to \cdots \tag{4}$$

where each $G_n(F_i) \simeq K_n(F_i)$ is a finite Abelian group for $n \geq 2$ — by [15, Theorem 7.1.12] or by Quillen's result. So, from Eq. 4 above, $G_n((F_i)_\alpha[T])$ is finite for all $n \geq 2$, i.e. $K_n((F_i)_\alpha[T]) \simeq G_n((F_i)_\alpha[T])$ is a finite Abelian group. Hence $(K_n(A'/JA')_\alpha[T])$, as a finite direct sum of Abelian groups of the form $K_n(F_i)_\alpha[T]$ is a finite group. Hence $K_n((A'/JA')_\alpha[T])\big(\frac{1}{s}\big)$ is torsion. So, for $A = B_\alpha(T)$ or $B'_\alpha[T]$, $K_n(A)\big(\frac{1}{s}\big) \simeq K_n\big((A/JA)\big(\frac{1}{s}\big)\big)$ is torsion and $\mathbb{Q} \otimes K_n(A)\big(\frac{1}{s}\big) = 0$.

So, by tensoring the Mayer-Vietoris exact sequence Eq. 3 with $\mathbb{Q}$ we get an isomorphism

$$\mathbb{Q} \otimes K_n(\Lambda_\alpha[T]) \simeq \mathbb{Q} \otimes K_n(\Gamma_\alpha[T])$$

for all $n \geq 2$. □

Theorem 4 *Let R, F, Λ, α; Γ, $\Lambda_\alpha[T]$, $\Gamma_\alpha[T]$ be as in Theorem 3. Let $\varphi_n : G_n(\Gamma_\alpha[T]) \longrightarrow G_n(\Lambda_\alpha[T])$ be the homomorphism induced by the exact functor $\mathcal{M}(\Gamma_\alpha[T]) \to \mathcal{M}(\Lambda_\alpha[T])$ given by 'restriction of scalars'. Then for all $n \geq 2$, φ_n has finite kernel and torsion cokernel and hence induces an isomorphism*

$$\mathbb{Q} \otimes G_n(\Gamma_\alpha[T]) \simeq \mathbb{Q} \otimes G_n(\Lambda_\alpha[T])$$

Proof First note that the exact functor $\mathcal{M}(\Gamma) \to \mathcal{M}(\Lambda)$ given by 'restriction of scalars' yields group homomorphisms $\delta_n : G_n(\Gamma) \to G_n(\Lambda)$. Now, by replacing the maximal order Γ in the proof of [15, Theorem 7.2.3, p. 146] or [16] with the α-invariant order Γ containing Λ, as in Theorem 2, we have that for all $n \geq 1$, $\delta_n : G_n(\Gamma) \to G_n(\Lambda)$ has finite kernel and cokernel. The proof in [15, Theorem 7.2.3] works for this Γ also. Now from [15, Theorem 7.5.3(b)] or [16], we have the following horizontal exact sequence and hence a commutative diagram

$$\begin{tikzcd}
G_n(\Gamma) \arrow[r, "1-\alpha_*"] \arrow[d, "\delta_n"] & G_n(\Gamma) \arrow[r] \arrow[d, "\delta_n"] & G_n(\Gamma_\alpha[T]) \arrow[r] \arrow[d, "\varphi_n"] & G_{n-1}(\Gamma) \arrow[r, "1-\alpha_*"] \arrow[d, "\delta_{n-1}"] & G_{n-1}(\Gamma) \arrow[d, "\delta_{n-1}"] \\
G_n(\Lambda) \arrow[r, "1-\alpha_*"] & G_n(\Lambda) \arrow[r] & G_n(\Lambda_\alpha[T]) \arrow[r] & G_{n-1}(\Lambda) \arrow[r, "1-\alpha_*"] & G_{n-1}(\Lambda)
\end{tikzcd} \tag{5}$$

By taking kernels and cokernels of vertical arrows in Eq. 5, we have a top (resp. bottom) horizontal exact sequence consisting of kernels (resp. cokernels) of the vertical maps. Since we saw above that δ_n has finite kernels and cokernels, we then have that $\phi_n : G_n(\Gamma_\alpha[T]) \to G_n(\Lambda_\alpha[T])$ has finite kernel and cokernel for each $n \geq 2$. Hence $\mathbb{Q} \otimes G_n(\Gamma_\alpha[T]) \simeq \mathbb{Q} \otimes G_n(\Lambda_\alpha[T])$. But $\Gamma_\alpha[T]$ is regular. Hence

$$\mathbb{Q} \otimes K_n(\Gamma_\alpha[T]) \simeq \mathbb{Q} \otimes G_n(\Lambda_\alpha[T]).$$ □

Theorem 5 *Let R, F, Σ, Λ, α, T be as in Theorem* 1. *Then for all $n \geq 2$, the map $\theta_n : G_n(\Lambda_\alpha[T]) \longrightarrow G_n(\Sigma_\alpha[T]) \simeq K_n(\Sigma_\alpha[T])$ induced by the canonical map $\Lambda_\alpha[T] \to \Sigma_\alpha[T]$ has finite kernel and torsion cokernel. Hence*

$$\mathbb{Q} \otimes G_n(\Lambda_\alpha[T]) \simeq \mathbb{Q} \otimes G_n(\Sigma_\alpha[T]) \simeq \mathbb{Q} \otimes K_n(\Sigma_\alpha[T]).$$

Proof Note that the canonical (inclusion) map $\Lambda \xrightarrow{\rho} \Sigma$ induces a group homomorphism $\rho_n : G_n(\Lambda) \to G_n(\Sigma) \simeq K_n(\Sigma)$ (note that $G_n(\Sigma) \simeq K_n(\Sigma)$ since Σ is regular).

Now, by [15, Theorem 7.5.3(b)] or [16], we have the following horizontal exact sequences and hence a commutative diagram

$$\begin{array}{ccccccccc}
G_n(\Lambda) & \xrightarrow{1-\alpha_*} & G_n(\Lambda) & \longrightarrow & G_n(\Lambda_\alpha[T]) & \longrightarrow & G_{n-1}(\Lambda) & \longrightarrow & G_{n-1}(\Lambda) \\
\downarrow{\scriptstyle \rho_n} & & \downarrow{\scriptstyle \rho_n} & & \downarrow{\scriptstyle \theta_n} & & \downarrow{\scriptstyle \rho_{n-1}} & & \downarrow{\scriptstyle \rho_{n-1}} \\
G_n(\Sigma) & \xrightarrow{1-\alpha_*} & G_n(\Sigma) & \longrightarrow & G_n(\Sigma_\alpha[T]) & \longrightarrow & G_{n-1}(\Sigma) & \longrightarrow & G_{n-1}(\Sigma)
\end{array} \tag{6}$$

Now, from the commutative diagram

$$\begin{tikzcd}
G_n(\Lambda) \arrow[rr, "\rho_n"] & & G_n(\Sigma) \simeq K_n(\Sigma) \\
& K_n(\Gamma) \arrow[ul, "\delta_n"] \arrow[ur, "\beta_n"'] &
\end{tikzcd} \tag{7}$$

we have

$$0 \to \ker \delta_n \to \ker \beta_n \to \ker \rho_n \to \operatorname{coker} \delta_n \to \operatorname{coker} \beta_n \to \operatorname{coker} \rho_n \to 0$$

Now, by the proof of Theorem 4, $\ker \delta_n$ and $\operatorname{coker} \delta_n$ are finite. Also by [15, Theorem 7.2.2] or [12], $\ker \beta_n$ is finite and $\operatorname{coker} \beta_n$ is torsion for all $n \geq 2$. Hence from diagram Eq. 7 above, $\ker \rho_n$ is finite and $\operatorname{coker} \rho_n$ is torsion for all $n \geq 2$. It then follows from the diagram Eq. 6 above that $\ker \theta_n$ is finite and $\operatorname{coker} \theta_n$ is torsion. □

Proof of Theorem 1 (a) From [15, Theorem 7.5.3(b)] or [16], we have an exact sequence

$$G_n(\Lambda) \xrightarrow{1-\alpha_*} G_n(\Lambda) \longrightarrow G_n(\Lambda_\alpha[T]) \longrightarrow G_{n-1}(\Lambda) \xrightarrow{1-\alpha_*} G_{n-1}(\Lambda)$$

Also by [15, Theorem 7.1.13] or [10] $G_n(\Lambda)$ is a finitely generated Abelian group for all $n \geq 1$. Hence $G_n(\Lambda_\alpha[T])$ is finitely generated for all $n \geq 2$. (b) That $\mathbb{Q} \otimes K_n(\Lambda_\alpha[T]) \simeq \mathbb{Q} \otimes G_n(\Lambda_\alpha[T])$ follow from Theorem 2 i.e. $\mathbb{Q} \otimes K_n(\Lambda_\alpha[T]) \simeq \mathbb{Q} \otimes K_n(\Gamma_\alpha[T])$ and Theorem 3 i.e. $\mathbb{Q} \otimes G_n(\Lambda_\alpha[T]) \simeq \mathbb{Q} \otimes K_n(\Sigma_\alpha[T])$. □

Remark 1 Since by Theorem 1(a), $G_n(\Lambda_\alpha[T])$ is finitely generated Abelian group for all $n \geq 2$, it follows that $K_n(\Lambda_\alpha[T])$ and $K_n(\Sigma_\alpha[T])$ have finite torsion free rank just like $G_n(\Lambda_\alpha[T])$.

Hence if $V = G \rtimes_\alpha T$ is a virtually infinite cyclic group, then $K_n(RV)$, $K_n(FV)$ have finite torsion-free rank for $n \geq 2$.

2.2. Finiteness of $G_{2(m+1)}(\Lambda_\alpha[T])$

In this subsection, we investigate under what circumstances $G_n(\Lambda_\alpha[T])$ could actually be a finite group. We prove below (see Theorem 6) that if F is a totally real field, then the group $G_{2(m+1)}(\Lambda_\alpha[T])$ is finite for all odd positive integers m. We state this formally:

Theorem 6 *Let R be the ring of integers in a totally real number field F, Λ an R-order in a semi-simple F-algebra, $\alpha : \Lambda \to \Lambda$ and R-automorsphim. Then for all odd positive integers m, $G_{2(m+1)}(\Lambda_\alpha[T])$ is a finite group. Hence in the notation of Theorem* 1, *$G_{2(m+1)}(RV)$ is finite.*

The proof of Theorem 6 will make use of the following:

Theorem 7 *Let F be a number field with ring of integers R, Λ and R-order in a semi-simple F-algebra Σ. Then* (*a*) *For all $n \geq 1$, $G_{2n}(\Lambda)$ is a finite group.* (*b*) *If F is totally real, then $G_{2m+1}(\Lambda)$ is also finite for all odd $m \geq 1$.*

Proof Part (a) is proved in [15] and [14]. See [15, Theorem 7.2.7].

If F is a totally real number field with ring of integers O_F, a similar proof works. We only have to show that $K_{2m+1}(\Gamma)$ is finite if Γ is a maximal order in a central division algebra D over a totally real number field F with ring of integer O_F. Let the dimension of D over F be s^2. We know from [15, Theorem 7.1.11] or [11] that $K_{2m+1}(\Gamma)$ is finitely generated. We only need to show that $K_{2m+1}(\Gamma)$ is torsion. Let $\mathrm{tr} : K_{2m+1}(\Gamma) \to K_{2m+1}(O_F)$ be the transfer map and $i : K_{2m+1}(O_F) \to K_{2m+1}(\Gamma)$ the map induced by the inclusion map $O_F \to \Gamma$. Let $x \in K_{2m+1}(\Gamma)$. Then $i \circ \mathrm{tr}(x) = x^{s^2}$. But $K_{2m+1}(\Gamma)$ is finite since it is also finitely generated. (See [2] for the proof that $K_{2m+1}(O_F)$ is torsion). □

Proof of Theorem 6 Assume that m is an odd positive integer. The we have an exact sequence

$$\cdots \to G_{2m+2}(\Lambda) \xrightarrow{1-\alpha_n} G_{2m+2}(\Lambda) \xrightarrow{\beta} G_{2m+2}(\Lambda_\alpha[T]) \xrightarrow{\gamma} G_{2m+1}(\Lambda) \to \cdots$$

where $G_{2m+2}(\Lambda)$ is finite by Theorem 7(a) and $G_{2m+1}(\Lambda)$ is finite by Theorem 7(b). So $G_{2m+2}(\Lambda_\alpha[T])/\operatorname{Im}\beta \simeq \operatorname{Im}\gamma$.

But $\operatorname{Im}\beta$ is finite and $\operatorname{Im}\gamma$ is also finite as a subgroup of the finite group $G_{2m+1}(\Lambda)$. Note that $\operatorname{Im}\beta$ is finite as a homomorphic image of the finite group $G_{2m+2}(\Lambda)$. Hence $G_{2m+2}(\Lambda_\alpha[T])$ is finite for all odd positive integers m. □

3. Mod-l^s and Profinite Higher K-theory of $\Lambda_\alpha(T)$

3.1. Mod-l^s Theory

3.1.1 Let $\mathcal{C}$ be an exact category, l a rational prime, s a positive integer, $M_{l^s}^{n+1}$ the $(n+1)$-dimensional mod-l^s-space, i.e. the space obtained from S^n by attaching an $(n+1)$-cell via a map of degree l^s (see [3, 15, 17]).

If X is an H-space, let $[M_{l^s}^{n+1}, X]$ be the set of homotopy classes of maps from $M_{l^s}^{n+1}$ to X.We shall write $\pi_{n+1}(X, \mathbb{Z}/l^s)$ for $[M_{l^s}^{n+1}, X]$. If $\mathcal{C}$ is an exact category and we put $X = BQ\mathcal{C}$, we write $K_n(\mathcal{C}, \mathbb{Z}/l^s)$ for $\pi_{n+1}(BQ\mathcal{C})$, we write $K_n(\mathcal{C}, \mathbb{Z}/l^s)$ for $\pi_{n+1}(\mathcal{C}, \mathbb{Z}/l^s)$ and $K_0\mathcal{C}, \mathbb{Z}/l^s$ for $K_0(\mathcal{C})\otimes\mathbb{Z}/l^s$. We shall refer to $K_n(\mathcal{C}, \mathbb{Z}/l^s)$ as mod-l^s K-theory of $\mathcal{C}$.

3.1.2 From [15, 8.1.2] or [13], we have an exact sequence

$$K_n(\mathcal{C}) \xrightarrow{l^s} K_n(\mathcal{C}) \xrightarrow{\rho} K_n\big(\mathcal{C}, \mathbb{Z}/l^s\big) \xrightarrow{\beta} K_{n-1}(\mathcal{C}) \longrightarrow K_{n-1}(\mathcal{C})$$

and hence a short exact sequence for all $n \geq 2$

$$0 \longrightarrow K_n(\mathcal{C})/l^s \longrightarrow K_n\big(\mathcal{C}, \mathbb{Z}/l^s\big) \longrightarrow K_n(\mathcal{C})[l^s] \longrightarrow 0$$

where $K_n(\mathcal{C})\big[l^s\big] = \big\{x \in K_n(\mathcal{C}) \mid l^s x = 0\big\}$.

Example 1

(i) Let A be a ring with identity and $\mathcal{P}(A)$ the category of finitely generated projective A-modules. We write $K_n(A, \mathbb{Z}/l^s)$ for $K_n(\mathcal{P}(A), \mathbb{Z}/l^s)$. We are interested in $A = \Lambda_\alpha(T)$. Note that $K_n(A, \mathbb{Z}/l^s)$ is also $\pi_n(BGL(A)^+, \mathbb{Z}/l^s)$.
(ii) Let A be a Noetherian ring and $\mathcal{M}(A)$ the category of finitely generated A-modules. We write $G_n(A, \mathbb{Z}/l^s)$ for $K_n(\mathcal{A}, \mathbb{Z}/l^s)$.
(iii) Let Y be a scheme, $\mathcal{C} = \mathcal{P}(Y)$ the category of locally free sheaves of O_X-modules of finite rank. We write $K_n(X, \mathbb{Z}/l^s)$ for $K_n(\mathcal{P}(Y), \mathbb{Z}/l^s)$ and observe that for $Y = \mathrm{Spec}(A)$, A a commutative ring, we recover $K_n(A, \mathbb{Z}/l^s)$ as in (i).
(iv) Let Y be a Noetherian scheme and $\mathcal{M}(Y)$ the category of coherent sheaves of O_Y-modules. We write $G_n(Y, \mathbb{Z}/l^s)$ for $K_n(\mathcal{M}(Y'), \mathbb{Z}/l^s)$ and when $Y = \mathrm{Spec}(A)$, where A is commutative, then we recover $G_n(A, \mathbb{Z}/l^s)$ as in (ii) above.
(v) It follows from Section 3.1.2 that we have exact sequences

$$0 \longrightarrow K_n(\Lambda_\alpha[T])/l^s \longrightarrow K_n\big(\Lambda_\alpha[T], \mathbb{Z}/l^s\big) \longrightarrow K_n(\Lambda_\alpha[T])[l^s] \longrightarrow 0$$

and

$$0 \longrightarrow G_n(\Lambda_\alpha[T])/l^s \longrightarrow G_n(\Lambda_\alpha[T], \mathbb{Z}/l^s) \longrightarrow G_n(\Lambda_\alpha[T])[l^s] \longrightarrow 0$$

3.2. Profinite Higher K-theory

3.2.1 Let $\mathcal{C}$ be an exact category, l a rational prime, s a positive integer $M_{l^\infty}^{n+1} = \varprojlim M_{l^s}^{n+1}$. We define the profinite K-theory of $\mathcal{C}$ by $K_n^{\mathrm{pr}}\big(\mathcal{C}, \hat{\mathbb{Z}}_l\big) := \big[M_{l^\infty}^{n+1}, BQ\mathcal{C}\big]$. We write $K_n\big(\mathcal{C}, \hat{\mathbb{Z}}_l\big)$ for $\varprojlim_s K_n\big(\mathcal{C}, \mathbb{Z}/l^s\big)$.

For more details on these constructions and their properties, see [15, Chapter 8] or [13].

Example 2

(i) For $\mathcal{C} = \mathcal{P}(A)$ as in Example 1(i), we shall write $K_n^{\mathrm{pr}}\big(A, \hat{\mathbb{Z}}_l\big)$ for $K_n^{\mathrm{pr}}\big(\mathcal{P}(A), \hat{\mathbb{Z}}_l\big)$ and $K_n\big(A, \hat{\mathbb{Z}}_l\big)$ for $K_n\big(\mathcal{P}(A), \hat{\mathbb{Z}}_l\big)$.
(ii) For $\mathcal{C} = \mathcal{M}(A)$ as in Example 1(ii), we shall write $G_n^{\mathrm{pr}}\big(A, \hat{\mathbb{Z}}_l\big)$ for $K_n^{\mathrm{pr}}\big(\mathcal{M}(A), \hat{\mathbb{Z}}_l\big)$ and $G_n\big(A, \hat{\mathbb{Z}}_l\big)$ for $K_n\big(\mathcal{M}(A), \hat{\mathbb{Z}}_l\big)$.
(iii) For $\mathcal{C} = \mathcal{P}(Y)$ as in Example 1(iii) we shall write $K_n^{\mathrm{pr}}\big(Y, \hat{\mathbb{Z}}_l\big)$ for $K_n^{\mathrm{pr}}\big(\mathcal{P}(Y), \hat{\mathbb{Z}}_l\big)$ and $K_n\big(Y, \hat{\mathbb{Z}}_l\big)$ for $K_n\big(\mathcal{P}(Y), \hat{\mathbb{Z}}_l\big)$.
(iv) For $\mathcal{C} = \mathcal{M}(Y)$ as in Example 1(iv), we shall write $G_n^{\mathrm{pr}}\big(Y, \hat{\mathbb{Z}}_l\big)$ for $K_n^{\mathrm{pr}}\big(Y, \hat{\mathbb{Z}}_l\big)$ and $G_n\big(Y, \hat{\mathbb{Z}}_l\big) = K_n\big(\mathcal{M}(Y), \hat{\mathbb{Z}}_l\big)$.

Remark 2 From the results obtained earlier by this author for general exact categories, (see [15, Chapter 8] or [13]) we can already deduce the following for $\mathcal{P}(\Lambda_\alpha[T])$ and $\mathcal{M}(\Lambda_\alpha[T])$.

(i) From [15, Lemma 8.2.1], we have the following exact sequences for $n \geq 1$.

$$\text{(a)}\ \ 0 \longrightarrow \varprojlim_s{}^1 K_{n+1}\big(\Lambda_\alpha[T], \mathbb{Z}/l^s\big) \longrightarrow K_n^{\mathrm{pr}}\big(\Lambda_\alpha[T], \hat{\mathbb{Z}}_l\big) \longrightarrow K_n\big(\Lambda_\alpha[T], \hat{\mathbb{Z}}_l\big) \longrightarrow 0$$

$$\text{(b)}\ \ 0 \longrightarrow \varprojlim_s{}^1 G_{n+1}\big(\Lambda_\alpha[T], \mathbb{Z}/l^s\big) \longrightarrow G_n^{\mathrm{pr}}\big(\Lambda_\alpha[T], \hat{\mathbb{Z}}_l\big) \longrightarrow G_n\big(\Lambda_\alpha[T], \hat{\mathbb{Z}}_l\big) \longrightarrow 0.$$

(ii) From [15, Theorem 8.2.2] we have for all $n \geq 2$,

$$\text{(a)}\ \ \varprojlim_s K_n^{\mathrm{pr}}\big(\Lambda_\alpha[T], \hat{\mathbb{Z}}_l\big)\big[l^s\big] = 0; \quad \varprojlim_s{}^1 K_{n+1}\big(\Lambda_\alpha[T], \mathbb{Z}/l^s\big) = \operatorname{div} K_n^{\mathrm{pr}}\big(\Lambda_\alpha[T], \hat{\mathbb{Z}}_l\big);$$

$$\text{(b)}\ \ \varprojlim_s G_n^{\mathrm{pr}}\big(\Lambda_\alpha[T], \hat{\mathbb{Z}}_l\big)\big[l^s\big] = 0; \quad \varprojlim_s{}^1 G_{n+1}\big(\Lambda_\alpha[T], \mathbb{Z}/l^s\big) = \operatorname{div} G_n^{\mathrm{pr}}\big(\Lambda_\alpha[T], \hat{\mathbb{Z}}_l\big).$$

(iii) From [15, Lemma 8.2.2] or [13], we have

$$(a)\quad \varprojlim_s K_n^{\mathrm{pr}}\big(\Lambda_\alpha[T], \hat{\mathbb{Z}}_l\big)/l^s \simeq K_n\big(\Lambda_\alpha[T], \hat{\mathbb{Z}}_l\big);$$

$$(b)\quad \varprojlim_s G_n^{\mathrm{pr}}\big(\Lambda_\alpha[T], \hat{\mathbb{Z}}_l\big)/l^s \simeq G_n\big(\Lambda_\alpha[T], \hat{\mathbb{Z}}_l\big).$$

3.3. Some Computations

3.3.1 The aim of this subsection is to prove Theorem 8 below. Before stating the result, we first explain the construction of map φ in Theorem 8(c) below.

Note that for any exact category $\mathcal{C}$, the natural map $M_{l^\infty}^{n+1} \to S^{n+1}$ induces a map

$$\big[S^{n+1}, BQ\mathcal{C}\big] \xrightarrow{\varphi} \big[M_{l^\infty}^{n+1}, BQ\mathcal{C}\big], \quad \text{i.e.,}$$

$$K_n(\mathcal{C}) \xrightarrow{\varphi} K_n^{\mathrm{pr}}\big(\mathcal{C}, \hat{\mathbb{Z}}_l\big).$$

So when $\mathcal{C} = \mathcal{M}\big(\Lambda_\alpha[T]\big)$ we have a map

$$\varphi : G_n(\Lambda_\alpha[T]) \longrightarrow G_n^{\mathrm{pr}}\big(\Lambda_\alpha[T], \hat{\mathbb{Z}}_l\big).$$

Theorem 8 *Let R be the ring of integers in a number field F, Λ any R-order in a semi-simple F-algebra Σ, $\alpha : \Lambda \to \Lambda$ an R-automorphism of Λ, $\Lambda_\alpha[T]$ the α-twisted Laurent series ring over Λ. Then, for all $n \geq 2$:*

(a) $\operatorname{div} G_n^{\mathrm{pr}}\big(\Lambda_\alpha[T], \hat{\mathbb{Z}}_l\big) = 0$.
(b) $G_n^{\mathrm{pr}}\big(\Lambda_\alpha[T], \hat{\mathbb{Z}}_l\big) \simeq G_n\big(\Lambda_\alpha[T], \hat{\mathbb{Z}}_l\big)$ *is an l-complete profinite Abelian group.*
(c) *The map $G_n\big(\Lambda_\alpha[T]\big) \longrightarrow G_n^{\mathrm{pr}}\big(\Lambda_\alpha[T], \hat{\mathbb{Z}}_l\big)$ is injective with uniquely l-divisible cokernel.*

Proof (a) From Remark 2(ii)(b), we have

$$\varprojlim_s{}^1 G_{n+1}\big(\Lambda_\alpha[T], \mathbb{Z}/l^s\big) = \operatorname{div} G_n^{\mathrm{pr}}\big(\Lambda_\alpha[T], \hat{\mathbb{Z}}_l\big), \tag{8}$$

for all $n \geq 2$. Now, by Theorem 1(a) $G_n(\Lambda_\alpha[T])$ is finitely generated for all $n \geq 1$. Hence $G_n(\Lambda_\alpha[T], \mathbb{Z}/l^s)$ is finite for all $n \geq 1$. In particular, $G_{n+1}(\Lambda_\alpha[T], \mathbb{Z}/l^s)$ is finite for all $n \geq 2$ and so $\varprojlim_s G_{n+1}(\Lambda_\alpha[T], \mathbb{Z}/l^s) = 0$ for all $n \geq 2$. Hence from Eq. 8, $\operatorname{div} G_n^{\mathrm{pr}}(\Lambda_\alpha[T], \hat{\mathbb{Z}}_l) = 0$ for all $n \geq 2$.

(b) We saw in (a) above that $G_n(\Lambda_\alpha[T], \mathbb{Z}/l^s)$ is a finite group for all $n \geq 1$. Hence in the exact sequence

$$0 \longrightarrow \varprojlim_s{}^1 G_{n+1}(\Lambda_\alpha[T], \mathbb{Z}/l^s) \longrightarrow G_n^{\mathrm{pr}}\big(\Lambda_\alpha[T], \hat{\mathbb{Z}}_l\big) \longrightarrow G_n\big(\Lambda_\alpha[T], \hat{\mathbb{Z}}_l\big) \longrightarrow 0$$

we have $\varprojlim{}^1 G_{n+1}(\Lambda_\alpha[T], \mathbb{Z}/l^s) = 0$. Hence,

$$G_n^{\mathrm{pr}}\big(\Lambda_\alpha[T], \hat{\mathbb{Z}}_l\big) \simeq G_n\big(\Lambda_\alpha[T], \hat{\mathbb{Z}}_l\big). \tag{9}$$

Now, by Remark 2(ii)(b),

$$G_n^{\mathrm{pr}}\big(\Lambda_\alpha[T], \hat{\mathbb{Z}}_l\big)/l^s \simeq G_n\big(\Lambda_\alpha[T], \hat{\mathbb{Z}}_l\big). \tag{10}$$

So, from Eqs. 9 and 10 $G_n^{\mathrm{pr}}(\Lambda_\alpha[T], \hat{\mathbb{Z}}_l)/l^s \simeq G_n^{\mathrm{pr}}(\Lambda_\alpha[T], \hat{\mathbb{Z}}_l)$ i.e. $G_n^{\mathrm{pr}}(\Lambda_\alpha[T], \hat{\mathbb{Z}}_l) \simeq G_n(\Lambda_\alpha[T], \hat{\mathbb{Z}}_l)$ is l-complete. It is profinite since $G_n(\Lambda_\alpha[T], \hat{\mathbb{Z}}_l) = \varprojlim G_n(\Lambda_\alpha[T], \mathbb{Z}/l^s)$ where each $G_n(\Lambda_\alpha[T], \mathbb{Z}/l^s)$ is a finite group.

(c) Since for all $n \geq 1$, $G_n(\Lambda_\alpha[T])$ is a finitely generated Abelian group (see 2.1.1(a)), it follows that $G_n(\Lambda_\alpha[T])$ is a finite group for each n. Hence $G_n(\Lambda_\alpha[T])_l$ has no non-trivial divisible subgroups. Hence by [15, Corollary 8.2.1] or [13], kernel and cokernel of φ are uniquely l-divisible. But $G_n(\Lambda_\alpha[T])$ is finitely generated and so, $\ker\phi = \operatorname{div}\ker\phi = 0$, as subgroups of $G_n(\Lambda_\alpha[T])$. □

4. $K_{-1}(\Lambda)$, $K_{-1}(\Lambda_\alpha[T])$, Λ Arbitrary Orders

4.1. Finite Generation of $K_{-1}(\Lambda)$, $K_{-1}(\Lambda_\alpha[T])$.

Let R be the ring of integers in a number field F, Λ any R-order in a semi-simple F-algebra Σ, $\alpha : \Lambda \to \Lambda$ and R-automorphism of Λ, $\Lambda_\alpha[T]$, the α-twisted Laurent polynomial ring over Λ. We prove in this section that $K_{-1}(\Lambda)$ and $K_{-1}(\Lambda_\alpha[T])$ are finitely generated Abelian groups for arbitrary R-orders Λ in semi-simple F-algebras. Note that the proof in [9] by Farrell/Jones is for $\Lambda = \mathbb{Z}G$, G a finite group.

Also D. Carter shows in [4] that $K_{-1}(RG)$ is finitely generated and here we show that this result also holds more generally for arbitrary orders.

Finally we prove also that $NK_{-1}(\Lambda, \alpha) = 0$ and so, $K_{-1}(\Lambda_\alpha[t]) \simeq K_{-1}(\Lambda)$.

Theorem 9 *Let F be an algebraic number field with ring of integers R, Λ any R-order in a semi-simple F-algebra Σ, $\alpha : \Lambda \to \Lambda$ an R-automorphism of Λ, $\Lambda_\alpha[T]$ the α-twisted Laurent series ring over Λ. Then*

(a) $K_{-1}(\Lambda)$ *is a finitely generated Abelian group.*
(b) $K_{-1}(\Lambda_\alpha[T])$ *is a finitely generated Abelian group.*
(c) $K_{-1}(\Lambda) \simeq K_{-1}(\Lambda_\alpha[t])$.

Proof (a) Let Γ be a maximal R-order containing Λ. Then, there exists a positive integer s such that $s\Gamma \subset \Lambda$. Then $\underline{q} = s\Gamma$ is an ideal of Λ and Γ. Put $B = \Lambda/\underline{q}$, $B' = \Gamma/\underline{q}$. Then we have a cartesian square

$$\begin{array}{ccc} \Lambda & \longrightarrow & \Gamma \\ \downarrow & & \downarrow \\ B & \longrightarrow & B' \end{array}$$

and hence a Mayer-Vietoris sequence

$$\begin{aligned} \cdots \to K_1(B') \longrightarrow K_0(\Lambda) \longrightarrow K_0(\Gamma) \oplus K_0(B) \longrightarrow K_0(B') \\ \longrightarrow K_{-1}(\Lambda) \longrightarrow K_{-1}(\Gamma) \oplus K_{-1}(B) \to \cdots \end{aligned} \tag{11}$$

Now by [1, Prop. 10.1, p. 685], $K_{-i}(A) = 0$ for $i \geq 1$ and any quasi-regular ring A. Note that B, B' are finite rings and hence quasi-regular. Also Γ is quasi-regular. Hence for $A = B$, B' or Γ, $K_{-i}(A) = 0$ for $i \geq 1$. So the sequence Eq. 11 becomes

$$\cdots \longrightarrow K_0(\Lambda) \longrightarrow K_0(\Gamma) \oplus K_0(B) \longrightarrow K_0(B') \longrightarrow K_{-1}(\Lambda) \longrightarrow 0. \tag{12}$$

To show that $K_{-1}(\Lambda)$ is finitely generated it suffices from Eq. 12 to show that $K_0(B')$ is finitely generated. Now B' is a finite Artinian ring and so, by [1, p. 465], $K_0(B') \simeq K_0(B'/JB')$ where JB' = radical of B'. But B'/JB' is a finite semi-simple ring and so, $K_0(B') \simeq K_0(B'/JB')$ is a finite direct sum of K_0 of (finite) fields each of which is isomorphic to $\mathbb{Z}$. Hence $K_0(B')$ is a (free) Abelian group of finite rank and hence is finitely generated. Hence $K_{-1}(\Lambda)$ is finitely generated.

(b) Let Γ be an α-invariant order containing Λ as in Corollary 1. Let s be a positive integer such that $s\Gamma \subset \Lambda$ and put $\underline{q} = s\Gamma$, $B = \Lambda/\underline{q}$, $B' = \Gamma/\underline{q}$. Then we have cartesian squares

$$\begin{array}{ccc} \Lambda & \longrightarrow & \Gamma \\ \downarrow & & \downarrow \\ B & \longrightarrow & B' \end{array} \tag{13}$$

and

$$\begin{array}{ccc} \Lambda_\alpha[T] & \longrightarrow & \Gamma_\alpha[T] \\ \downarrow & & \downarrow \\ B_\alpha[T] & \longrightarrow & B'_\alpha[T] \end{array} \tag{14}$$

and hence a Mayer-Vietoris sequence

$$\begin{aligned} \cdots \longrightarrow K_0(\Lambda_\alpha[T]) \longrightarrow K_0(\Gamma_\alpha[T]) \oplus K_0(B_\alpha[T]) \\ \longrightarrow K_0(B'_\alpha[T]) \longrightarrow K_{-1}(\Lambda_\alpha[T]) \longrightarrow 0. \end{aligned} \tag{15}$$

where $\Gamma_\alpha[T]$, $B_\alpha[T]$ and $B'_\alpha[T]$ are quasi-regular (see [9]). If $A = \Gamma_\alpha[T]$, $B_\alpha[T]$ or $B'_\alpha[T]$ and T^n is the free Abelian group of rank n. Then by [1, Prop. 10.1], $K_{-i}(A) = 0$ for $i \geq 1$.

Also, by Serre's theorem $K_0(A) \to K_0(A[T^n])$ is an epimorphism (see [7]). Since $K_{-n}(A)$ is a direct summand of the cokernel of $K_0(A) \to K_0(A[T^n])$ we have $K_{-n}(A) = 0$ for $n \geq 1$. So from the exact sequence Eq. 11, we have $K_{-n}(\Lambda_\alpha[T]) = 0$ for $n \geq 2$ and $K_0(B'_\alpha[T]) \longrightarrow K_{-1}(\Lambda_\alpha[T])$ is an epimorphism.

By mapping the Mayer-Vietoris sequence associated with the cartesian square Eq. 11 to the Mayer-Vietoris sequence associated with square Eq. 12, we have a commutative square

$$\begin{array}{ccc} K_0(B') & \longrightarrow & K_{-1}(\Lambda) \\ \downarrow & & \downarrow \\ K_0(B'_\alpha[T]) & \longrightarrow & K_{-1}(\Lambda_\alpha[T]). \end{array} \tag{16}$$

To prove that $K_{-1}(\Lambda) \longrightarrow K_{-1}(\Lambda_\alpha[T])$ is an epimorphism, it suffices to prove that $K_0(B') \longrightarrow K_0(B'_\alpha[T])$ is an epimorphism in the commutative diagram

$$\begin{array}{ccc} K_0(B') & \longrightarrow & K_0(B'_\alpha[T]) \\ \downarrow & & \downarrow \\ K_0(B'/JB') & \longrightarrow & K_0((B'/JB')_\alpha[T]) \end{array}$$

where the vertical maps are isomorphisms. Also by [7, Theorem 27], the map $K_0(B'/JB') \longrightarrow K_0((B'/JB')_\alpha[T])$ is an epimorphism. Hence $K_0(B') \longrightarrow K_0(B'_\alpha[T])$ is an epimorphism. So $K_{-1}(\Lambda) \longrightarrow K_{-1}(\Lambda_\alpha[T])$ is an epimorphism. Since by (a), $K_{-1}(\Lambda)$ is finitely generated, then $K_{-1}(\Lambda_\alpha[T])$ is also finitely generated.

(c) By definition, $K_{-1}(\Lambda_\alpha[t]) \simeq K_{-1}(\Lambda) \oplus NK_{-1}(\Lambda, \alpha)$. So it suffices to show that $NK_{-1}(\Lambda, \alpha) = 0$.

Let Λ, Γ, $B = \Lambda/\underline{q}$, $B' = \Gamma/\underline{q}$ be as in the proof of (a) (b). Then we have two cartesian squares

$$\begin{array}{ccc} \Lambda & \longrightarrow & \Gamma \\ \downarrow & & \downarrow \\ B & \longrightarrow & B' \end{array} \tag{17}$$

and

$$\begin{array}{ccc} \Lambda_\alpha[t] & \longrightarrow & \Gamma_\alpha[t] \\ \downarrow & & \downarrow \\ B_\alpha[t] & \longrightarrow & B'_\alpha[t] \end{array} \tag{18}$$

where $\Gamma_\alpha[t]$, $B_\alpha[t]$ and $B'_\alpha[t]$ are quasi-regular as well as Γ, B, B'. Hence we have Mayer-Vietoris sequences

$$\cdots \to K_0(\Lambda_\alpha[t]) \longrightarrow K_0(\Gamma_\alpha[t]) \oplus K_0(B_\alpha[t]) \longrightarrow K_0\big(B'_\alpha[t]\big) \longrightarrow K_{-1}(\Lambda_\alpha[t]) \to \ldots \tag{19}$$

and

$$\cdots \to K_0(\Lambda) \longrightarrow K_0(\Gamma) \longrightarrow K_0(B) \longrightarrow K_0(B') \longrightarrow K_{-1}(\Lambda) \to \cdots \tag{20}$$

where for $A = \Gamma, B, B', \Gamma_\alpha[t], B_\alpha[t], B'_\alpha[t]$, $K_{-i}(A) = 0$ for $i \geq 1$ (see [1, Prop. 10.1]). By mapping Eqs. 19 to 20 and taking kernels, we have that

$$NK_{-1}(\Lambda, \alpha) = \operatorname{coker}(NK_0(\Gamma, \alpha) \oplus NK_0(B, \alpha) \longrightarrow NK_0(B', \alpha)).$$

So it suffices to show that $NK_0(B', \alpha) = 0$. Since B', $B'_\alpha[t]$ are quasi-regular, the result follows from [6, Lemma 2.4]. So $NK_{-1}(\Lambda, \alpha)=0$ and hence $K_{-1}(\Lambda[t]) \simeq K_{-1}(\Lambda)$. □

Corollary 2 *let R be the ring of integers in a number field F, $V = G \rtimes_\alpha T$ a virtually infinite cyclic group where G is a finite group and the action of the infinite cyclic group T on G is given by $\alpha(g) = tgt^{-1}$ for all $g \in G$. Then $K_{-1}(RV)$ is a finitely generated Abelian group.*

Corollary 3 *Let α be an automorphism of a finite group G, R the ring of integers in a number field F. Denote the induced automorphism on RG also by α. Then $K_{-1}(RG) \simeq K_{-1}((RG)_\alpha[t])$ is a finitely generated Abelian group.*

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References

1. Bass, H.: Algebraic K-theory. W.A. Benjamin, Menlo Park (1968)
2. Borel, A.: Stable real cohomology of arithmetic groups. Ann. Sci. École Norm. Sup. **7**(4), 235–272 (1984)
3. Browder, W.: Algebraic K-theory with Coefficients $\mathbb{Z}/p$. Lecture Notes in Mathematics, vol. 657, pp. 40–84. Springer, Berlin (1978)
4. Carter, D.W.: Localization in lower algebraic K-theory. Comm. Algebra **8**(7), 603–622 (1980)
5. Charney, R.: A Note on Excision in K-theory. Lecture Notes in Mathematics, vol. 1046, pp. 47–48. Springer, Berlin (1984)
6. Conolly, F., Prassidis, S.: On the exponents of NK groups of virtually infinite cyclic groups. Car. Math. Bull. **45**(2), 180–195 (2002)
7. Farrell, F.T., Hsiang, W.C.: A formula for $K_1(R\alpha[T])$. Applications of Categorical Algebra. Proceedings of Symposia in Pure Mathematics, vol. 17, pp. 192–218. American Mathematical Society, Providence (1970)

8. Farrell, F.T., Jones, L.E.: Isomorphisms conjectures in algebraic K-theory. J. Amer. Math. Soc. **6**, 249–297 (1993)
9. Farrell, F.T., Jones, L.E.: The lower algebraic K-theory of virtually infinite cyclic groups. K-Theory **9**, 13–30 (1995)
10. Kuku, A.O. K-theory of grouprings of finite groups over maximal orders in division algebras. J. Algebra **91**(1), 18–31 (1985)
11. Kuku, A.O.: K_n, SK_n of integral group rings and orders. Contemp. Math. AMS **55**, 333–338 (1986)
12. Kuku, A.O.: Ranks of K_n and G_n of orders and grouprings of finite groups over integers in number fields. J. Pure Appl. Algebra **138**, 39–44 (1999)
13. Kuku, A.O.: Profinite and continuous higher K-theory of exact categories, orders and group rings. K-theory **22**, 367–392 (2001)
14. Kuku, A.O.: Finiteness of higher K-groups of orders and group rings. K-Theory **36**, 51–58 (2005)
15. Kuku, A.O.: Representation Theory and Higher Algebraic K-theory. Chapman and Hall, London (2007)
16. Kuku, A.O., Tang, G.: Higher K-theory of grouprings of virtually infinite cyclic groups. Math. Ann. **323**, 711–725 (2003)
17. Neisendorfer, J.: Primary homotopy theory. Mem. Amer. Math. Soc. **232**, (1980)
18. Reiner, I.: Maximal Orders. Academic Press, London (1975)
19. Weibel, C.: Mayer-Vietoris sequences and mod-p-K-Theory. Lecture Notes in Mathematics, vol. 966, pp. 390–407. Springer, Berlin (1982)